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New Algorithm for Attitude Determination Using Global Positioning System Signals

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A novel technique for finding a point-by-point (deterministic) attitude solution of a vehicle using Global Positioning System phase difference measurements is presented. This technique transforms a general cost function into a more numerically efficient form by determining three-dimensional vectors in either the body or reference coordinate system. Covariance relationships for the new algorithm, as well as methods that minimize the general cost function, are also derived. The equivalence of the general cost function and transformed cost function is shown for the case of orthogonal baselines or sight lines. Simulation results are shown that demonstrate the usefulness of the new algorithm and covariance expressions.

Introduction

THE utilization of phase difference measurements from Global Positioning System (GPS) receivers provides a novel approach for three-axis attitude determination and/or estimation. These measurements have been successfully used to determine the attitude of both aircraft[1] and spacecraft.[2,3] Recently, much attention has been placed on spacecraft-based applications. One of the first space-based GPS experiments for attitude determination was flown on the radar calibration (RADCAL) spacecraft.[4] To obtain maximum GPS visibility and to reduce signal interference due to multipath reflection, GPS patch antennas were placed on the top surface of the spacecraft bus. Although the antenna baselines were short for attitude determination, accuracies between 0.5–1.0 deg (root mean square) were achieved.

In this paper, the problem of finding the attitude from GPS phase difference measurements using deterministic approaches is addressed. Error sources, such as integer sign ambiguity,[5] are not investigated. These errors are assumed to be accounted for before the attitude determination problem is solved. The most common GPS attitude determination scheme minimizes a cost function constituting the sum weighted two-norm residuals between the measured and determined phase difference quantities. However, as of this writing, the optimal attitude solution that minimizes this general cost function can only be found using iterative techniques, such as gradient search methods. A suboptimal solution involves transforming the general cost function into a form that can be minimized without iterative intense methods. One such technique, developed by Cohen,[1] transforms the general cost function into a form identical to Wahba's problem.[6] Therefore, fast algorithms such as QUEST[7] and FOAM[8] can then be used to determine the attitude. Cohen showed that the solution based on Wahba's problem is almost an order of magnitude faster than a conventional nonlinear least-squares algorithm.

Cohen's approach involves a two-step process. The first step involves finding a weighting matrix, using a singular-value decomposition (SVD), which transforms the baseline configuration to an equivalent orthonormal basis. At least three noncoplanar baselines must exist to perform this transformation. If this is not the case, the transformation can still be accomplished as long as three noncoplanar sight lines exist. However, an SVD must be performed for each new sight line, which can be computationally expensive, whereas the baseline transformation has to be done only once. The second step involves finding the attitude using the fast algorithms such as QUEST or FOAM. Because the weighting matrix transforms the baseline configuration to an equivalent orthonormal basis, suboptimal attitude solutions may arise if the baseline configuration does not already form an orthonormal basis.[1] An example of this scenario is when three baselines are coplanar. To determine the optimal attitude, iterative techniques that minimize the general cost function must be used. The method presented in this paper also is suboptimal for the case where the baseline (or sight line) configuration does not form an orthonormal basis. However, it does not require an SVD of a 3×3 matrix to perform the orthonormal transformation.

Bar-Itzhack et al.[9] show another analytical conversion of the basic GPS scalar difference measurements into unit vectors to be used in Wahba's problem. This is accomplished by expressing the angle determined by one of the baselines, which describes a cone around the baseline vector, and likewise for the second baseline, into a three-dimensional vector resolved in a reference coordinate system. Attitude solutions are provided for baselines that constitute Cartesian and non-Cartesian coordinate systems; however, these solutions shown in Ref. 9 involve only two baseline vectors. This paper generalizes these results to multiple baseline vectors. Also, covariance relations are shown for the new approach, as well as for techniques that minimize the general cost function directly. This allows users to quantify any additional errors produced by transforming the general cost function into Wahba's form.

The organization of this paper proceeds as follows. First, the concept of the GPS phase difference measurement is introduced. Then, the general cost function used for GPS-based attitude determination is reviewed. Next, Cohen's method for transforming the general cost function into Wahba's problem is shown. Also, system observability using two baselines is discussed. Then, a general technique for transforming the general cost function is developed. Also, the equivalence of the general and transformed (Wahba) cost functions for orthogonal baselines and/or sight lines is discussed. Next, a covariance analysis is performed on the new algorithm and on algorithms that minimize the general cost function directly. Finally, results are shown for a simulated vehicle with near-orthogonal baselines, nonorthogonal baselines, and baselines that are nearly collinear.

Background

In this section, a brief background of the GPS phase difference measurement is shown. The GPS constellation of spacecraft was developed for accurate navigation information of land-based, air, and spacecraft user systems. Spacecraft applications initially involved obtaining accurate orbit information and accurate time-tagging of

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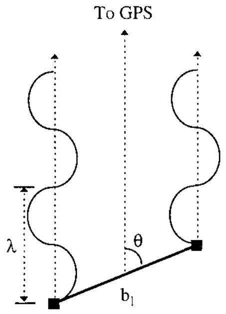

Fig. 1 GPS wavelength and wave front angle.

spacecraft operations. However, attitude determination of vehicles, such as spacecraft or aircraft, has gained much attention. The main measurement used for attitude determination is the phase difference of the GPS signal received from two antennas separated by a baseline. The principle of the wave front angle and wavelength, which are used to develop a phase difference, is illustrated in Fig. 1.

The phase difference measurement is obtained by

$$b_l \cos\theta = \lambda(n + \Delta\phi^0/2\pi) \tag{1}$$

where b_l is the baseline length, θ is the angle between the baseline and the line of sight to the GPS spacecraft, n is the number of integer wavelengths between two receivers, $\Delta\phi^0$ is the actual phase difference measurement, and λ is the wavelength of the GPS signal. The two GPS frequency carriers are L1 at 1575.42 MHz and L2 at 1227.6 MHz. Then, assuming no integer offset, we define a normalized phase difference measurement $\Delta\phi$ by

$$\Delta\phi \equiv \frac{\lambda\Delta\phi^0}{2\pi b_l} = \boldsymbol{b}^T A \boldsymbol{s} \tag{2}$$

where $\boldsymbol{s} \in R^3$ is the normalized line of sight vector to the GPS spacecraft in an inertial frame, $\boldsymbol{b} \in R^3$ is the normalized baseline vector, which is the relative position vector from one antenna to another, and $A \in SO(3)$ is the attitude matrix, which is a Lie group of orthogonal matrices with determinant 1, i.e., $A^TA = I$ and $\det A = 1$.

Cohen's Method

In this section, Cohen's method[1] for determining the attitude of a vehicle using Eq. (2) is reviewed. The general cost function to be minimized is given by

$$J(A) = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij}\left(\Delta\phi_{ij} - \boldsymbol{b}_i^T A \boldsymbol{s}_j\right)^2 \tag{3}$$

where m represents the number of baselines and n represents the number of observed GPS spacecraft. The parameters w_{ij} serve to weight each individual phase measurement. The phase measurement can contain noise, which is modeled by

$$\Delta\phi_{ij} = \Delta\phi_{ij}^{\text{true}} + v_{ij} \tag{4}$$

where v_{ij} is a zero-mean stationary Gaussian process with covariance given by σ_{ij}^2. The maximum likelihood estimate for w_{ij} is given by $1/\sigma_{ij}^2$. If the weights w_{ij} factor into a satellite-dependent and a baseline-dependent factor, i.e., $w_{ij} = w_{bi}w_{sj}$, then the cost function in Eq. (3) may be rewritten as[1]

$$J(A) = \left\| W_B^{\frac{1}{2}}(\Delta\Phi - B^TAS)W_S^{\frac{1}{2}} \right\|_F^2 \tag{5}$$

where $\| \|_F^2$ denotes the Frobenius norm and

$$\Delta\Phi = \begin{bmatrix} \Delta\phi_{11} & \Delta\phi_{12} & \cdots & \Delta\phi_{1n} \\ \Delta\phi_{21} & & & \\ & & \ddots & \vdots \\ \Delta\phi_{m1} & & \cdots & \Delta\phi_{mn} \end{bmatrix} \tag{6a}$$

$$B = [\boldsymbol{b}_1 \,\vdots\, \boldsymbol{b}_2 \,\vdots\, \cdots \,\vdots\, \boldsymbol{b}_m] \tag{6b}$$

$$S = [\boldsymbol{s}_1 \,\vdots\, \boldsymbol{s}_2 \,\vdots\, \cdots \,\vdots\, \boldsymbol{s}_n] \tag{6c}$$

The weighting matrices W_B and W_S are applicable to the rows (baselines) and columns (spacecraft) of $\Delta\Phi$, respectively.

If the quaternion[10] representation is used for the attitude matrix, then Eq. (5) leads to a quartic dependence in the quaternions. In Wahba's problem, this dependence cancels out of the cost function. To cancel this dependence in Eq. (5), Cohen chooses the following weighting matrix for W_B:

$$W_B = V_B \Sigma_B^{-2} V_B^T \tag{7}$$

where V_B and Σ_B are given from an SVD of B, i.e., $B = U_B \Sigma_B V_B^T$. From Eq. (7), the matrix B must be full rank, which means that at least three noncoplanar baselines must be used. However, if this is not true, a solution can still be found as long as three noncoplanar sight lines exist. This can be accomplished by performing an SVD of S and choosing W_S as in the same form in Eq. (7). However, an SVD must be performed for each sight line. This is more computationally expensive than using Eq. (7), which may be done once for constant baselines. It is also not obvious that Eq. (7) is consistent with Eq. (3). Substituting Eq. (7) into the general cost function in Eq. (5) leads to Wahba's problem, which maximizes

$$J'(A) = \text{tr}\left(ASW_S\Delta\Phi^T W_B B^T\right) \equiv \text{tr}(AG^T) \tag{8}$$

However, by constraining W_B or W_S, the solution using the transformed cost function in Eq. (8) is suboptimal for nonorthogonal baselines or sightlines. The concept of the suboptimal solution will be discussed in detail later.

Attitude Determination from Vectorized Measurements

In this section, a new method for attitude determination from GPS phase measurements is developed. This new method extends the method shown in Ref. 9, which converts the phase measurements into vector measurements. The general method for the vectorized measurements is based on an algorithm given by Shuster.[11] Also, a covariance analysis is performed for the new method and for methods that minimize the general cost function in Eq. (3) directly.

The vectorized measurement problem involves determining the sight line vector in the body frame, denoted by $\tilde{\boldsymbol{s}}_j \equiv A\boldsymbol{s}_j$, or the baseline in an inertial frame, denoted by $\bar{\boldsymbol{b}}_i \equiv A^T\boldsymbol{b}_i$. For the sight line case, the following cost function is minimized:

$$J_j(\tilde{\boldsymbol{s}}_j) = \frac{1}{2}\sum_{i=1}^{m}\frac{1}{\sigma_{ij}^2}\left(\Delta\phi_{ij} - \boldsymbol{b}_i^T\tilde{\boldsymbol{s}}_j\right)^2 \quad \text{for} \quad j = 1, 2, \ldots, n \tag{9}$$

The minimization of Eq. (9) is straightforward and leads to[11]

$$\tilde{\boldsymbol{s}}_j = M_j^{-1}\boldsymbol{y}_j \tag{10}$$

where

$$M_j = \sum_{i=1}^{m}\frac{1}{\sigma_{ij}^2}\boldsymbol{b}_i\boldsymbol{b}_i^T \quad \text{for} \quad j = 1, 2, \ldots, n \tag{11a}$$

$$\boldsymbol{y}_j = \sum_{i=1}^{m}\frac{1}{\sigma_{ij}^2}\Delta\phi_{ij}\boldsymbol{b}_i \quad \text{for} \quad j = 1, 2, \ldots, n \tag{11b}$$

The error covariance of $\tilde{\boldsymbol{s}}_j$ is given by

$$P_j = M_j^{-1} \tag{12}$$

If the sight line in the body is required to be normalized, then the cost function in Eq. (9) must be minimized subject to the constraint $\tilde{\boldsymbol{s}}^T\tilde{\boldsymbol{s}} = 1$. However, Shuster[11] showed that the error introduced by ignoring this constraint is on the order of $m^{-1}\text{tr}[|\tilde{\boldsymbol{s}}_j|^{-2}(I - \tilde{\boldsymbol{s}}_j\tilde{\boldsymbol{s}}_j^T) \times P_j(I - \tilde{\boldsymbol{s}}_j\tilde{\boldsymbol{s}}_j^T)]$, which is usually negligible. The solution of Wahba's problem as shown later will determine the optimal attitude (in the least-squares sense), which results in a normalized body vector. Therefore, the normalization constraint may be ignored. Also, the normalized error covariance is singular, as shown in Ref. 11. This singularity is avoided by using the covariance given by Eq. (12). For a discussion of singularity issues for measurement covariances, see Ref. 12.

From Eq. (10) it is seen that at least three noncoplanar baselines are required to determine the sight lines in the body frame. This is analogous to the problem posed by Cohen.[1] However, if only two noncollinear baselines exist, a solution is again possible as long as three noncoplanar sight lines exist. This approach determines the baselines in the inertial frame, using the following cost function:

$$J_i(\bar{\boldsymbol{b}}_i) = \frac{1}{2}\sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\left(\Delta\phi_{ij} - \bar{\boldsymbol{b}}_i^T \boldsymbol{s}_j\right)^2 \quad \text{for} \quad i = 1, 2, \ldots, m \tag{13}$$

The minimization of Eq. (13) is again straightforward and leads to

$$\bar{\boldsymbol{b}}_i = N_i^{-1}\boldsymbol{z}_i \tag{14}$$

where

$$N_i = \sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\boldsymbol{s}_j\boldsymbol{s}_j^T \qquad \text{for} \qquad i = 1, 2, \ldots, m \tag{15a}$$

$$\boldsymbol{z}_i = \sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\Delta\phi_{ij}\boldsymbol{s}_j \qquad \text{for} \qquad i = 1, 2, \ldots, m \tag{15b}$$

The error covariance of $\bar{\boldsymbol{b}}_i$ is given by

$$Q_i = N_i^{-1} \tag{16}$$

The case with two noncollinear baselines and two noncollinear sight lines can also be solved for either the baseline inertial case or sight line body case. Solving for the latter case yields

$$\tilde{\boldsymbol{s}}_j = a_{1j}\boldsymbol{b}_1 + a_{2j}\boldsymbol{b}_2 + a_{3j}(\boldsymbol{b}_1 \times \boldsymbol{b}_2) \qquad \text{for} \qquad j = 1, 2 \tag{17}$$

where

$$a_{1j} = |\boldsymbol{b}_1 \times \boldsymbol{b}_2|^{-2}[\Delta\phi_{1j} - \Delta\phi_{2j}(\boldsymbol{b}_1 \cdot \boldsymbol{b}_2)] \quad \text{for} \quad j = 1, 2 \tag{18a}$$

$$a_{2j} = |\boldsymbol{b}_1 \times \boldsymbol{b}_2|^{-2}[\Delta\phi_{2j} - \Delta\phi_{1j}(\boldsymbol{b}_1 \cdot \boldsymbol{b}_2)] \quad \text{for} \quad j = 1, 2 \tag{18b}$$

$$\begin{aligned} a_{3j} &= \pm|\boldsymbol{b}_1 \times \boldsymbol{b}_2|^{-2}\Big\{f_j|\boldsymbol{b}_1 \times \boldsymbol{b}_2|^2 - \Delta\phi_{1j}^2 \\ &\quad + 2\Delta\phi_{1j}\Delta\phi_{2j}(\boldsymbol{b}_1 \cdot \boldsymbol{b}_2) - \Delta\phi_{2j}^2\Big\}^{\frac{1}{2}} \quad \text{for} \quad j = 1, 2 \end{aligned} \tag{18c}$$

where $f_j = |\tilde{\boldsymbol{s}}_j|^2$. Equation (18c) involves knowledge of $|\tilde{\boldsymbol{s}}_j|^2$. However, this quantity can be assumed to be 1 with reasonable accuracy. Also, from Eq. (18c), there are two possible solutions for the body sight lines. However, this sign ambiguity can usually be resolved from the geometry of the vehicle to the GPS spacecraft. The error covariance is given by[11]

$$P_j = TL_jT^T \tag{19}$$

where

$$T = [\boldsymbol{b}_1 \;\vdots\; \boldsymbol{b}_2 \;\vdots\; \boldsymbol{b}_1 \times \boldsymbol{b}_2] \tag{20a}$$

$$L_j = \left[\begin{array}{c|c} D_j & \boldsymbol{l}_j \\ \hline \boldsymbol{l}_j^T & d_j \end{array}\right] \tag{20b}$$

and

$$D_j = UP_{\phi_j}U \tag{21a}$$

$$U = |\boldsymbol{b}_1 \times \boldsymbol{b}_2|^{-2}\begin{bmatrix} 1 & -\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 \\ -\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 & 1 \end{bmatrix} \tag{21b}$$

$$P_{\phi_j} = \begin{bmatrix} \sigma_{1j}^2 & 0 \\ 0 & \sigma_{2j}^2 \end{bmatrix} \tag{21c}$$

$$\boldsymbol{l}_j = \mp|\boldsymbol{b}_1 \times \boldsymbol{b}_2|^{-1}\left[1 - \boldsymbol{\psi}_j^T D_j \boldsymbol{\psi}_j\right]^{-\frac{1}{2}} UP_{\phi_j}U\boldsymbol{\psi}_j \tag{21d}$$

$$d_j = |\boldsymbol{b}_1 \times \boldsymbol{b}_2|^{-2}\left[1 - \boldsymbol{\psi}_j^T D_j \boldsymbol{\psi}_j\right]^{-1}\boldsymbol{\psi}_j^T UP_{\phi_j}U\boldsymbol{\psi}_j \tag{21e}$$

$$\boldsymbol{\psi}_j \equiv \begin{bmatrix} \Delta\phi_{1j} \\ \Delta\phi_{2j} \end{bmatrix} \tag{21f}$$

The covariance in Eq. (19) is singular. However, this does not affect the determination of the attitude error covariance, as will be shown. Also, the method can be trivially modified to determine the baselines in inertial space.

Attitude Determination

The attitude determination problem using body sight lines is very similar to that using inertial baselines, and so we may consider only the former case. The attitude is determined by using the following cost function:

$$J(A) = \frac{1}{2}\sum_{j=1}^{n}(\tilde{\boldsymbol{s}}_j - A\boldsymbol{s}_j)^T M_j(\tilde{\boldsymbol{s}}_j - A\boldsymbol{s}_j) \tag{22}$$

This cost function is not identical to Wahba's problem because the quartic dependence in the quaternion does not cancel, unless the baselines form an orthonormal basis so that M_j is given by a scalar times the identity matrix. The cost function in Eq. (22) is in fact equivalent to the general cost function in Eq. (3). This is shown by substituting Eqs. (10) and (11) into Eq. (22) and expanding terms, giving

$$J(A) = \frac{1}{2}\sum_{j=1}^{n}\left(\boldsymbol{y}_j^T M_j^{-1}\boldsymbol{y}_j - 2\boldsymbol{y}_j^T A\boldsymbol{s}_j + \boldsymbol{s}_j^T A^T M_j A\boldsymbol{s}_j\right) \tag{23}$$

Expanding Eq. (23) now yields

$$\begin{aligned} J(A) &= \frac{1}{2}\sum_{j=1}^{n}\left(\boldsymbol{y}_j^T M_j^{-1}\boldsymbol{y}_j - \sum_{i=1}^{m}\frac{1}{\sigma_{ij}^2}\Delta\phi_{ij}^2\right) \\ &\quad + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{\sigma_{ij}^2}\left(\Delta\phi_{ij} - \boldsymbol{b}_i^T A\boldsymbol{s}_j\right)^2 \end{aligned} \tag{24}$$

Because the first term in Eq. (24) is independent of attitude, it is clear that this cost function is equivalent to the general cost function in Eq. (3). To reduce the cost function in Eq. (22) into a form corresponding to Wahba's problem, the condition that M_j is given by a scalar times the identity matrix must be valid. Therefore, if the baselines do not form an orthonormal basis, then the attitude solution is suboptimal.

Attitude Covariance

Wahba posed the three-axis determination problem in terms of finding the proper orthogonal attitude matrix that minimizes the least-squares cost function, given by

$$J(A) = \frac{1}{2}\sum_{j=1}^{n}a_j|\tilde{\boldsymbol{s}}_j - A\boldsymbol{s}_j|^2 \tag{25}$$

Several efficient algorithms have been developed to solve this problem, e.g., QUEST[7] and FOAM.[8] Another solution for the attitude matrix is given by performing an SVD of the following matrix:

$$F = \sum_{j=1}^{n}a_j\tilde{\boldsymbol{s}}_j\boldsymbol{s}_j^T = U\Sigma V^T \tag{26}$$

The optimal solution for the attitude matrix is given by[13]

$$A_{\text{opt}} = U_+ V_+^T \tag{27}$$

where

$$U_+ = U[\text{diag}(1, 1, \det U)] \tag{28a}$$

$$V_+ = V[\text{diag}(1, 1, \det V)] \tag{28b}$$

The convariance of the estimation error angle vector in the body frame is given by[13]

$$\begin{aligned} E\{\boldsymbol{\delta\alpha\delta\alpha}^T\} &= P_{\text{body}} = \left(I - FA_{\text{opt}}^T\right)^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}a_i a_j \\ &\quad \times [\tilde{\boldsymbol{s}}_j\times]E\left\{\boldsymbol{e}_i\boldsymbol{e}_j^T\right\}[\tilde{\boldsymbol{s}}_j\times]^T\left(I - FA_{\text{opt}}^T\right)^{-1} \end{aligned} \tag{29}$$

where $[\tilde{s}_j\times]$ represents the cross-product matrix (see Ref. 13) and $\delta\alpha$ is the small error angle and where, for any k,

$$e_k \equiv A_{\text{opt}}\delta s_k - \delta\tilde{s}_k \tag{30}$$

The terms δs and $\delta\tilde{s}$ represent variations in the inertial and body sight lines, respectively. The expectation in Eq. (29) can be written as

$$E\left\{e_i e_j^T\right\} = A_{\text{opt}} E\left\{\delta s_i \delta s_j^T\right\} A_{\text{opt}}^T + E\left\{\delta\tilde{s}_i \delta\tilde{s}_j^T\right\} \tag{31}$$

Assuming that the only errors are in the effective phase measurements reduces Eq. (29) to

$$P_{\text{body}} = \left(I - FA_{\text{opt}}^T\right)^{-1} \sum_{j=1}^{n} a_j^2 [\tilde{s}_j\times] P_j [\tilde{s}_j\times]^T \left(I - FA_{\text{opt}}^T\right)^{-1} \tag{32}$$

Now using the approximation of

$$\tilde{s}_j \approx A_{\text{opt}} s_j \tag{33}$$

yields

$$I - FA_{\text{opt}}^T \approx \sum_{j=1}^{n} a_j \left(I - \tilde{s}_j \tilde{s}_j^T\right) = \sum_{j=1}^{n} a_j [\tilde{s}_j\times][\tilde{s}_j\times]^T \equiv X \tag{34}$$

and thus the error angle covariance is given by

$$P_{\text{body}} \approx X^{-1} \left\{ \sum_{j=1}^{n} a_j^2 [\tilde{s}_j\times] P_j [\tilde{s}_j\times]^T \right\} X^{-1} \tag{35}$$

Note that, if the covariances P_j are multiples of the identity, $P_j = \sigma_j^2 I$, and then setting $a_j = \sigma_j^{-2}$ would yield

$$P_{\text{body}} \approx \left[\sum_{j=1}^{n} \sigma_j^{-2} [\tilde{s}_j\times][\tilde{s}_j\times]^T \right]^{-1} = X^{-1} \tag{36}$$

Therefore, in this case the covariance in Eq. (36) would be identical to the covariance given by QUEST.[7] The best suboptimal weighting factor a_j in Eq. (35) can be found by minimizing the trace of P_{body}. However, this is extremely complex. If Eq. (36) is still a good approximation, then a_j can be chosen to minimize some matrix norm of the following:

$$J(a_j) = \|a_j P_j - I\| \tag{37}$$

An alternative to Eq. (37) is to minimize the following cost function for some matrix norm:

$$J(a_j) = \left\|a_j I - P_j^{-1}\right\| \tag{38}$$

For example, minimizing Eq. (38) with a Frobenius norm results in

$$a_j = \tfrac{1}{3}\text{tr}\left(P_j^{-1}\right) = \tfrac{1}{3}\text{tr}(M_j) \tag{39}$$

Once a proper weight is determined, then Wahba's problem in Eq. (25) can be solved. The covariance of the attitude errors is given by Eq. (35).

Transforming the general cost function in Eq. (3) results in a suboptimal solution. To quantify the errors introduced by the suboptimal solution, the error attitude covariance for the general cost function is derived. This is accomplished by using results from maximum likelihood estimation.[14] The Fisher information matrix for a parameter vector x is given by

$$F_{xx} = E\left\{\frac{\partial}{\partial x \partial x^T} J(x)\right\}_{x_{\text{true}}} \tag{40}$$

where $J(x)$ is the negative log likelihood function, which is the loss function in this case. If the measurements are Gaussian and linear in the parameter vector, then the error covariance is given by

$$P_{xx} = F_{xx}^{-1} \tag{41}$$

Because the cost function in Eq. (22) is equivalent to the full cost function in Eq. (3), Eq. (22) can be used to determine the covariance of the optimal solution. First, the attitude matrix is approximated by

$$A = e^{-[\delta\alpha\times]} A_{\text{true}} \approx \left(I - [\delta\alpha\times] + \tfrac{1}{2}[\delta\alpha\times]^2\right) A_{\text{true}} \tag{42}$$

Equations (42) and (33) are next substituted into Eqs. (22) and (40) to determine the Fisher information matrix. First-order terms vanish in the partials, and third-order terms become zero because $E\{\delta\alpha\} = \mathbf{0}$. Also, assuming that the quartic terms are negligible leads to the following simple form for the optimal covariance:

$$P_{\text{opt}} \approx \left[\sum_{j=1}^{n} [\tilde{s}_j\times] P_j^{-1} [\tilde{s}_j\times]^T \right]^{-1} \tag{43}$$

Note that the optimal covariance in Eq. (43) reduces to the covariance in Eq. (36) if the condition $P_j = \sigma_j^2 I$ is true. The errors introduced when using a suboptimal solution can now be compared with the expected performance of minimizing the general cost function in Eq. (3). Also, for the case of two baselines and two sight lines, the optimal covariance can be derived by using the cost function in Eq. (3) in the Fisher information matrix, which leads to

$$P_{\text{opt}} \approx \left[\sum_{i=1}^{2} \sum_{j=1}^{2} \frac{1}{\sigma_{ij}^2} (\tilde{s}_j \times b_i)(\tilde{s}_j \times b_i)^T \right]^{-1} \tag{44}$$

The covariance analysis can be easily extended to the case where the baselines in inertial space are determined. The body covariance for the transformed cost function in this case becomes

$$P_{\text{body}} \approx \left[\sum_{i=1}^{m} a_i [b_i\times]^2 \right]^{-1} \sum_{i=1}^{m} a_i^2 [b_i\times] A Q_i A^T [b_i\times]^T \times \left[\sum_{i=1}^{m} a_i [b_i\times]^2 \right]^{-1} \tag{45}$$

The error covariance for the optimal solution is given by

$$P_{\text{opt}} \approx \left[\sum_{i=1}^{m} [b_i\times] A Q_i^{-1} A^T [b_i\times]^T \right]^{-1} \tag{46}$$

Simulation Results

In this section, simulation results are shown using the new algorithm and covariance expressions. Three case are presented. The first case involves three baselines that are nearly orthogonal. The second involves three baselines that do not constitute an orthogonal set. The third case involves three baselines, where the first two baselines are far from constituting an orthogonal set, i.e., nearly collinear. Although the third case would most likely never be used in a practical application, it provides a radical test comparison between the optimal and suboptimal solutions. It is assumed that the vehicle is always in the view of two GPS spacecraft with constant and normalized sight lines given by

$$s_1 = (1/\sqrt{3})[1 \quad 1 \quad 1]^T \qquad s_2 = (1/\sqrt{2})[0 \quad 1 \quad 1]^T \tag{47}$$

The three normalized baseline cases are given by the following.

Case 1:

$$b_1 = \frac{1}{\sqrt{1.09}}[1 \quad 0.3 \quad 0]^T \qquad b_2 = [0 \quad 1 \quad 0]^T \qquad b_3 = [0 \quad 0 \quad 1]^T \tag{48a}$$

Case 2:

$$b_1 = (1/\sqrt{2})[1 \quad 1 \quad 0]^T \qquad b_2 = [0 \quad 1 \quad 0]^T \qquad b_3 = [0 \quad 0 \quad 1]^T \tag{48b}$$

Case 3:

$$b_1 = \frac{1}{\sqrt{1.02}}[0.1 \quad 1 \quad 0.1]^T \qquad b_2 = [0 \quad 1 \quad 0]^T \qquad b_3 = [0 \quad 0 \quad 1]^T \tag{48c}$$

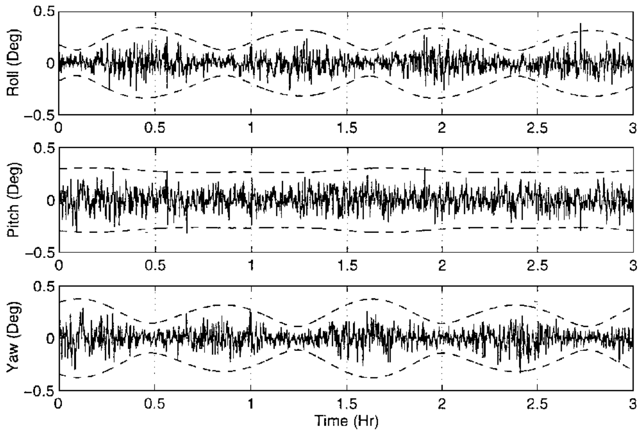

Fig. 2 Attitude errors and sigma bounds for case 1: suboptimal solution.

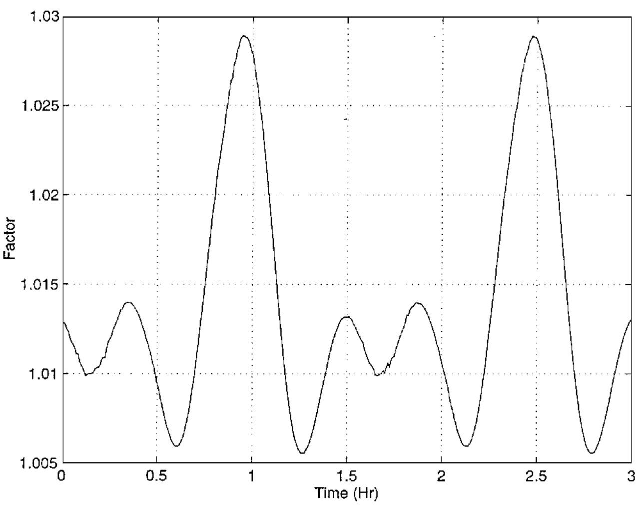

Fig. 3 Error factor for case 1: standard deviation factor between optimal and suboptimal solution.

The noise for each phase difference measurement is assumed to have a normalized standard deviation of $\sigma = 0.001$ (corresponding to an attitude error of about 0.5 deg). Also, the attitude of the vehicle is assumed to be Earth pointing with a rotation of 236 deg/h about the vehicle's y axis (negative orbit normal), while holding the remaining axis rotations to zero. The spacecraft's z axis is defined to be pointed nadir, and the x axis completes the triad.

If the baselines do not constitute an orthogonal set, the solution of the transformed cost function to Wahba's form is suboptimal. However, the covariance analysis shown in this paper can be used to assess the errors introduced from the transformation. All simulation results presented in the figures use a_j given by Eq. (39). Figure 2 shows the attitude errors and three-sigma bounds by solving Wahba's form for case 1. This shows the excellent agreement between theory and simulated measurement processes. To quantify the error introduced by using the suboptimal solution, the following error factor is used:

$$f = \frac{1}{m_{\text{tot}}} \sum_{k=1}^{m_{\text{tot}}} \frac{\operatorname{tr}\left\{\operatorname{diag}\left[P_{\text{body}}^{\frac{1}{2}}\right]\right\}}{\operatorname{tr}\left\{\operatorname{diag}\left[P_{\text{opt}}^{\frac{1}{2}}\right]\right\}} \tag{49}$$

where m_{tot} represents the total number of measurements used in the simulation. A plot of the error factor at each time is shown in Fig. 3. Equation (49) represents the average of the curve shown in Fig. 3. Clearly, the suboptimal solution is adequate, with a maximum error of about 3%. A plot of the standard deviation errors for the suboptimal and optimal solutions for case 2 is shown in Fig. 4. The optimal standard deviation error is always lower than the suboptimal solution. A plot of the error factor at each time is shown in Fig. 5.

Table 1 Weighting factor performance comparisons

	Case 1, f	Case 2, f	Case 3, f
$a_j = \dfrac{1}{(P_j)_{\max}}$	1.014	1.151	58.13
$a_j = \dfrac{2}{(P_j)_{\max} + (P_j)_{\min}}$	1.014	1.151	58.16
$a_j = \dfrac{\left(P_j^{-1}\right)_{\max} + \left(P_j^{-1}\right)_{\min}}{2}$	1.014	1.151	58.29
$a_j = \left[\dfrac{\operatorname{tr}(P_j)}{\operatorname{tr}\left(P_j^{-1}\right)}\right]^{\frac{1}{2}}$	1.014	1.151	58.43
$a_j = \frac{1}{3}\operatorname{tr}\left(P_j^{-1}\right)$	1.014	1.150	58.13

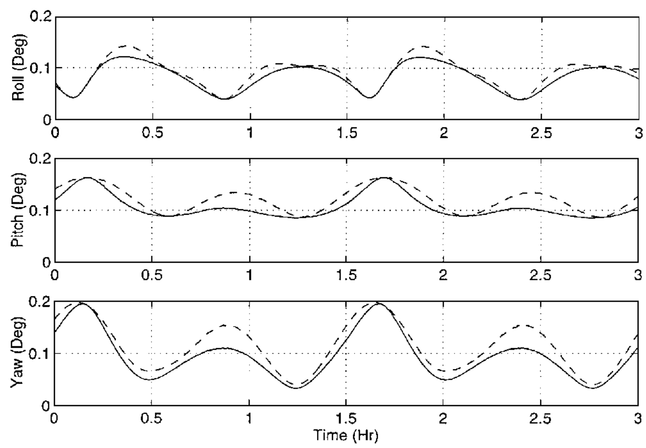

Fig. 4 Standard deviation comparison for case 2: ——, optimal and - - - -, suboptimal.

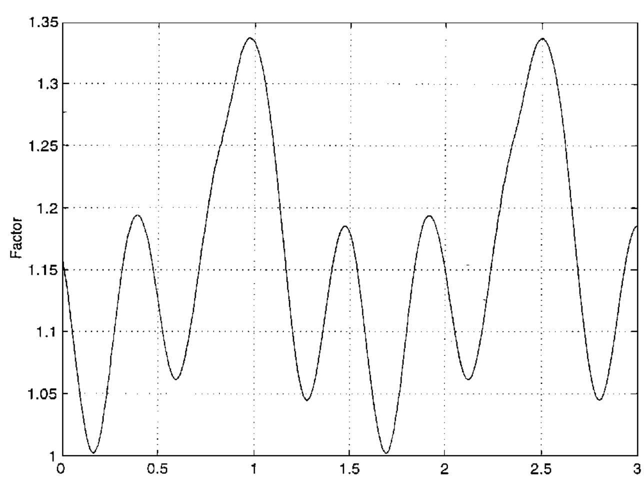

Fig. 5 Error factor for case 2: standard deviation factor between optimal and suboptimal solution.

For this case, the suboptimal solution can produce large errors, with a maximum error of about 35%. This is due to the nonorthogonal baselines and the attitude of the vehicle. Results using various values of a_j for all three case are shown in Table 1 (the subscripts max and min denote eigenvalues). Clearly, various choices for the weighting factors a_j do not affect system performance. Also, case 3, where the baselines are nearly collinear, results in a substantial degradation in performance when using the suboptimal solution as compared with the optimal solution. Therefore, the covariance analysis is extremely helpful for determining whether the suboptimal and/or the optimal solution meets required performance specifications.

Conclusions

The problem of determining the attitude of a vehicle using GPS phase measurements was addressed in this paper. A general method that transforms the general GPS cost function into a Wahba cost function was presented. Covariance equations for both the new method and methods that solve the general cost function were developed. It was shown that the transformation produces suboptimal attitude solutions for nonorthogonal baselines and sight lines. The equivalence of both covariance equations for orthogonal baselines and/or sight lines was also shown. Simulation results indicate that the new method is adequate for nearly orthogonal baselines or sight lines but can produce large errors for nearly collinear baselines or sight lines, as compared with methods that minimize the general cost function directly. This paper provides a means of accessing various performance criteria, such as computational efficiency vs attitude accuracy, for the particular application.

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References

[1]Cohen, C. E., "Attitude Determination Using GPS," Ph.D. Dissertation, Dept. of Aeronautics and Astronautics, Stanford Univ., Stanford, CA, Dec. 1992.

[2]Melvin, P. J., and Hope, A. S., "Satellite Attitude Determination with GPS," *Advances in the Astronautical Sciences*, Vol. 85, Pt. 1, 1993, pp. 59–78 (AAS Paper 93-556).

[3]Melvin, P. J., Ward, L. M., and Axelrad, P., "The Analysis of GPS Attitude Data from a Slowly Rotating, Symmetrical Gravity Gradient Satellite," *Advances in the Astronautical Sciences*, Vol. 89, Pt. 1, 1995, pp. 539–558 (AAS Paper 95-113).

[4]Lightsey, E. G., Cohen, C. E., Feess, W. A., and Parkinson, B. W., "Analysis of Spacecraft Attitude Measurements Using Onboard GPS," *Advances in the Astronautical Sciences*, Vol. 86, 1994, pp. 521–532 (AAS Paper 94-063).

[5]Cohen, C. E., and Parkinson, B. W., "Integer Ambiguity Resolution of the GPS Carrier for Spacecraft Attitude Determination," *Advances in the Astronautical Sciences*, Vol. 78, 1992, pp. 107–118 (AAS Paper 92-015).

[6]Wahba, G., "A Least Squares Estimate of Spacecraft Attitude," *SIAM Review*, Vol. 7, No. 3, 1965, p. 409.

[7]Shuster, M. D., and Oh, S. D., "Attitude Determination from Vector Observations," *Journal of Guidance and Control*, Vol. 4, No. 1, 1981, pp. 70–77.

[8]Markley, F. L., "Attitude Determination from Vector Observations: A Fast Optimal Matrix Algorithm," *Journal of the Astronautical Sciences*, Vol. 41, No. 2, 1993, pp. 261–280.

[9]Bar-Itzhack, I. Y., Montgomery, P. Y., and Garrick, J. C., "Algorithms for Attitude Determination Using GPS," *Proceedings of the AIAA Guidance, Navigation, and Control Conference* (New Orleans, LA), AIAA, Reston, VA, 1997 (AIAA Paper 97-3616).

[10]Shuster, M. D., "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993, pp. 439–517.

[11]Shuster, M. D., "Efficient Algorithms for Spin-Axis Attitude Estimation," *Journal of the Astronautical Sciences*, Vol. 31, No. 2, 1983, pp. 237–249.

[12]Shuster, M. D., "Kalman Filtering of Spacecraft Attitude and the QUEST Model," *Journal of the Astronautical Sciences*, Vol. 38, No. 3, 1990, pp. 377–393.

[13]Markley, F. L., "Attitude Determination Using Vector Observations and the Singular Value Decomposition," *Journal of the Astronautical Sciences*, Vol. 36, No. 3, 1988, pp. 245–258.

[14]Shuster, M. D., "Maximum Likelihood Estimation of Spacecraft Attitude," *Journal of the Astronautical Sciences*, Vol. 37, No. 1, 1989, pp. 79–88.